

Crystalline Beams*

Alessandro G. Ruggiero
Brookhaven National Laboratory

Abstract

Crystalline Beams are an *ordered* state of an ensemble of ions circulating in a Storage Ring with very small velocity fluctuations. They can be obtained from ordinary *warm* ion beams with the application of intense cooling techniques, namely electron and/or laser cooling. A phase transition occurs when sufficiently small velocity spreads are reached, freezing the particle-to-particle spacing in *strings*, *zig-zags*, and *helices*... The properties and feasibility of Crystalline Beams depend on the choice of the lattice of the Storage Ring. There are three issues closely related to the design of the Storage Ring, namely: the determination of Equilibrium Configurations, Confinement Conditions, and Stability Conditions. Of particular concern is the effect of the trajectory curvature and of the beam momentum spread. They both set the requirements on the amount of momentum cooling, on the focussing, and on the distribution of bending in the lattice of the Storage Ring. The practical demonstration of Crystalline Beams may create the basis for an advanced technology of particle accelerators. The limitations due to Coulomb intra-beam scattering and space-charge forces would be finally be brought under control, so that ordered beams of ions can be achieved for a variety of new applications.

Introduction

We should first differentiate between *Crystalline Structures* that have been observed experimentally, and *Crystalline Beams* that still have to be demonstrated. Crystalline Structures are stationary strongly Coulomb-coupled Plasmas, usually confined in ion traps. They are derived from non-neutral plasmas made of one gas component. After removing internal energy by means of laser cooling, the original ionized gas acquires a low internal temperature, as low as few μ Kelvin degrees. A transition of phase has been experimentally observed where ions acquire a rigid configuration spacing from each other at equal distance. The next section of this paper will describe properties of Crystalline Structures.

* Work performed under the auspices of the US Department of Energy

Crystalline Beams are an ordered state of an ensemble of ions, circulating in a storage ring, with very small velocity fluctuations. They have been studied theoretically but have never been observed experimentally in existing particle storage rings. The second section of this paper describes the properties of Crystalline Beams, and what are the conditions and requirements to be satisfied by both the ion beam and the storage ring.

Crystalline Beams can be obtained from ordinary warm ion beams with the application of fast and effective electron and/or laser cooling. Like in Crystalline Structures, a phase transition occurs when sufficiently small velocity spreads are reached, which freezes the ions to an ordered configuration, by equally spacing from each other. The simplest configuration is a *string* where the ions are equally spaced longitudinally from each other. More complex configurations are *zig-zags*, *helices*... which can always be described as several *strings* placed parallel to each other. The relevant parameter describing a *string* is the ion longitudinal spacing λ .

There are three steps involved to study the properties of Crystalline Beams. During the first step, one determines the Equilibrium Configuration, that is essentially how many parallel *strings* the whole Crystalline Beam is made of, and where it is located, namely the lattice of the storage ring. The second step introduces the Confinement Conditions, a set of equations that are to be satisfied by both the beam and the storage ring consistently with the postulated Configuration. Finally, in the third step, the stability of the Configuration is evaluated by solving another set of equations which takes into account the evolution of a small perturbation added to the system.

Differently from Crystalline Structures, of a particular concern for the stability of a Crystalline Beam is the effect of the trajectory curvature and of the beam momentum spread (*shear effect*). This sets very stringent requirements on the amount of momentum cooling, on the focussing, and on the distribution of bending in the lattice of the storage ring. One requirement is that the storage ring operates at energies well below the transition energy.

The requirements on the storage ring lattice are very stringent and not satisfied by ordinary operational storage rings, which therefore explains why Crystalline Beams have not been observed yet. A new concept of storage ring, the Circular Radio-Frequency Quadrupole Storage Ring (CRFQ), may have a lattice capable to satisfy the requirements and thus be ideal for the demonstration of Crystalline Beams.

In this paper we shall not describe the cooling process, the cooling requirements, and how the transition actually occurs. We shall limit ourselves to the study of the *ground state* of

a Crystalline Beam, namely its confined and stable configuration. The aim of the research is to demonstrate whether Crystalline Beams do indeed allow larger ion densities, and how to take advantage of the knowledge that the beam has an ordered configuration. As we shall see, in order to understand the formation of Crystalline Beams, it is necessary to observe the beam at microscopic level, that is at particle-to-particle distances, where Intra-Beam Scattering (Coulomb Scattering) and Space-Charge forces are unified. With this understood, it will be shown that the depression of the storage ring focussing caused by the space-charge forces, down to a half-integral structural resonance, determines the *range of existence* of a particular Crystalline Beam configuration. This eventually sets requirements on the storage ring periodicity and betatron tunes.

Crystalline Structures

As we have already said Crystalline Structures are stationary strongly Coulomb-coupled Plasmas which have been observed in Ion Traps. They are derived from one-component non-neutral plasmas after removing internal energy at sufficient high rate and effectively, with either Electron or Laser Cooling or the combination of both. Once a sufficient low temperature T has been reached, ions have tendency to equally distance from each other with a separation distance λ that depends on the density n . Moreover, ions vibrate at an angular frequency ω , around equally-spaced center of oscillations. As a consequence of the low temperature, ions have a residual vibration of amplitude $a \ll \lambda$.

In summary the main parameters describing a Crystalline Structure are:

- Ion Mass Number A
- Ion Charge State Q
- Ion Separation Distance λ
- Residual Vibration Amplitude a
- Vibration (angular) Frequency ω
- Residual Temperature T
- Ion Density n

Wigner's Crystal

It was Wigner to conceive the first Crystalline Structure in its simplest form [1]. Let us consider an ensemble of electrically charged particles, all identical to each other with charge state Q and mass number A . Let us assume that they are point-like with no internal structure, and that the only way they interact with each other is by Coulomb

interaction. Let us assume also that, as shown in Figure 1, all particles are at rest and are equally spaced, that is $T = a = 0$. Let λ be the particle-to-particle separation, the same in all three physical dimensions. Thus, the particles are located in such a way to form a three-dimensional grid with the lattice spacing λ . Moreover, let us assume that such crystalline structure extends to infinity in all three directions.

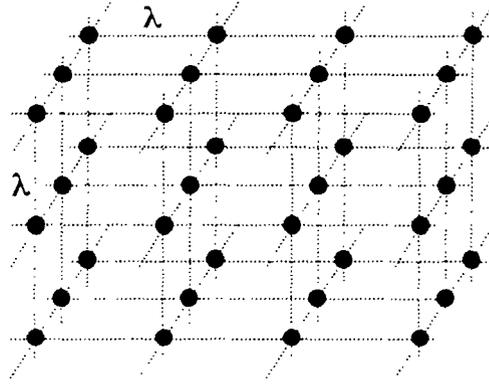


Figure 1. The Wigner's Crystal

It is obvious that this structure is an equilibrium configuration. Since particles do not move and are equally spaced, from symmetry, it is seen that there is no net force acting on any particle, and thus the particle will remain in that configuration forever, no matter how small the spacing λ . If we take any particle of this idealized Crystalline Beam and displace it by an amount a small compared to the spacing λ , leaving all other particles unperturbed, it will execute stable oscillations with amplitude a and angular frequency

$$\omega_0 = 2.4 Q^2 c^2 r_0 / A \lambda^3 \quad (1)$$

where c is the speed of light and $r_0 = 1.535 \times 10^{-18}$ m, the classical proton radius.

The Γ -Parameter

To each ion of a Crystalline Structure we can associate a Potential Energy U and a Kinetic Energy T

$$U = Q^2 e^2 / \lambda \quad (2)$$

$$T = m_0 A \omega^2 a^2 / 2 \quad (3)$$

where e is the electron charge and m_0 the proton mass at rest. In the early 60's Brush, Sahlin and Teller made several computer simulations with the Montecarlo approach of a

one-component plasma [2]. They took a number of identical ions interacting with each other immersed in a uniform neutralizing background. They found that the ratio

$$\Gamma = U/T \approx (\lambda/a)^2 \quad (4)$$

describes properties of an ensemble of electrically charged ions. $\Gamma < 1$ corresponds to a gas-like structure (as in Particle Accelerators), where particles move overtaking each other. By reducing the plasma internal temperature, or by bringing ions closer together, one obtains values $\Gamma \sim 1$ where an amorphous, fluid-like ordering appears. Particles acquire an oscillatory behavior with identifiable centers of oscillation; but the amplitudes are still large so that trajectories roll over each other. Reducing T and/or increasing U further to reach $\Gamma \gg 1$ finally yields to the Crystalline Structures. The amplitude a of the oscillation is now considerably smaller than the spacing λ . It was determined that the onset of the phase transition occurs at $\Gamma \sim 100$. From Eq. (4) we infer that a condition on the plasma temperature for Crystallization is $a < 0.1 \lambda$.

Similar computer simulations were done later by Rahman and Schiffer [3], using this time more efficient computers, and the Molecular Dynamics method introduced by Alder and Wainright [4] during the late 60's. The results by Rahman and Schiffer are in full agreement with those found earlier by Brush, Sahlin and Teller.

Experimental Observations

Crystalline Structures have been experimentally observed in Ion Traps; first, by H. Walther at the Max Planck Institute in Germany [5], and more recently in various other institutions [6].

The ion trap used by Walther was a simplified quadrupole storage ring with a diameter of only 11.5 cm. The toroidal vacuum chamber, with an inner diameter of 5 mm, was made of four circular electrodes parallel to each other and arranged to form the shape of a quadrupole, as shown in Figure 2. Rf voltage is applied between the electrodes to generate a radial parabolic potential in which to trap the ions. A neutral atomic pulse of ^{24}Mg was injected into the trap that was maintained at very good vacuum conditions. At the point of injection the atoms were partially stripped by a perpendicular beam of electrons. Because of their common electric charge, ions would repel each other, and distribute uniformly along the circumference of the storage ring; at the same time they are trapped transversely by the parabolic potential, and perform stable transverse oscillations.

A laser beam, tuned to the excitation frequency of the ions, shines along the longitudinal direction, causes momentum cooling, and the ions quickly pack toward the center of the distribution. Using the fluorescence emanated from the ions, once high densities and low temperature were reached, crystallization was observed, and several structures noticed: *strings*, *zig-zags*, and *helices*, as shown in Figure 3. Which configuration appeared depended on the density of the injected gas. The ion spacing measured was of the order of few tens of micron. The experimental observations were in full agreement with the theoretical and computer simulation predictions of the preceding years.

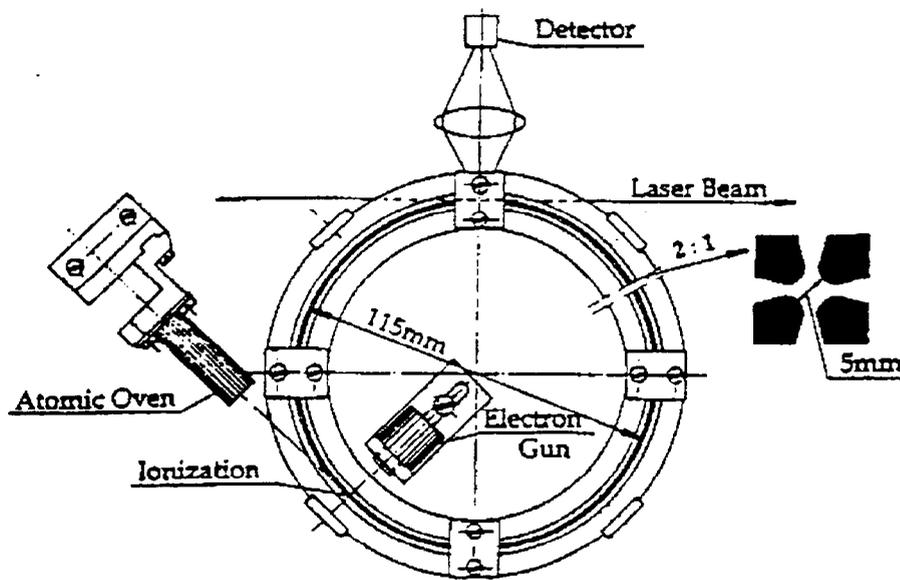


Figure 2. The Ion Trap used in Walther experiment [5].

In these experiments we always deal with stationary Structures and not with Beams, since, as in the case just described of Walther experiment, the ions in the trap were longitudinally at rest.

Crystalline Beams

The extrapolation of Crystalline Structures to Crystalline Beams is a natural aspiration. It is highly desirable to demonstrate that Crystalline Structures like those observed in Walther experiment can actually be made moving and circulating in a storage ring at at least non-relativistic velocities. The search for the formation of Crystalline Beams has now be going on for the last two decades mostly in European laboratories, like ASTRID in Copenaghen, TSR in Heidelberg, and CRYRING in Stockholm [7-9]. The search so far has remained fruitless. A storage ring CRYSTAL was even proposed by the Legnaro

Laboratory near Padua, Italy, just for the demonstration of Crystalline Beams. But once it was realized that the goal was difficult, if not impossible, to achieve, the project was discontinued. It was speculated that some type of organized beam formation was observed in NAP-M at Novosibirsky when the proton beam was cooled. But that remained just a speculation, since in the meantime, the storage ring was de-commissioned and dismantled.

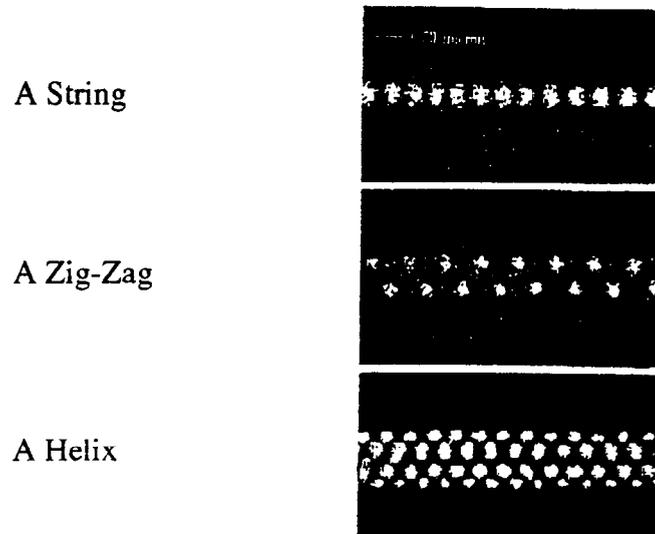


Figure 3. Observed Crystalline Structures in Walther experiment [5].

As we shall see, at most a *string* or a vertical *zig-zag* may be observed in available, conventional storage rings made of bending magnets and focussing quadrupoles. The requirements on the lattice are too severe and not easily satisfied by ordinary storage rings. Moreover, the *shear effect* caused by the bending and the curvature of the trajectory, that did not exist in the case of Crystalline Structures, complicates matter because of the tendency to *break* the Crystal. Crystalline Beams can be eventually demonstrated only with a novel concept of storage ring.

The Confinement Issue

The properties of the Wigner's Crystal do not change if the crystal is moving in any fixed direction at any constant velocity, apart from some trivial relativistic transformation. If the structure moves, we have than some sort of a Crystalline Beam. But it is an unrealistic beam, because its dimensions extend to infinity. A realistic beam has a finite transverse size. Particles at the edge of the beam will spread out and the crystalline structure will soon collapse. A Crystalline Beam can be maintained with a finite transverse size by applying external confining forces to balance the repulsion among particles. These forces

are those created by quadrupole magnets or other similar focussing elements that are used in ordinary particle storage rings.

The most common restoring forces are those that vary linearly with a particle transverse displacements from a reference trajectory. Hasse and Schiffer have studied this case with computer simulation [10]. They imposed on the beam a parabolic external potential of constant gradient along the main direction of motion, simultaneously focussing in both transverse dimensions. The motion of the particles is then oscillating at the angular frequency ω_b . Because typical phase-space trajectories are circular, they found configurations of Crystalline Beams which are not rectangular but take the shape notably of *strings*, *zig-zags*, *helices*, and any combination of them. Hasse and Schiffer used a parameter $\lambda_{HS} = (N/C) (3 m_0 A Q^2 e^2 / \omega_b^2)^{3/2}$ to determine when a particular configuration would appear, as shown in Figure 4. N is the total number of ions circulating in the storage ring of circumference C . The transition from one lower configuration to another is obtained by increasing the value of the parameter λ_{HS} , either by increasing the beam intensity N , or by reducing the strength of the external restoring forces.

The focussing model used by Hasse and Schiffer is unrealistic since it is made of a continuous constant gradient transport focussing simultaneously in both transverse directions. This can be realized only in a type of Betatron, a weak-focussing storage ring. Moreover the bending of the trajectory was completely ignored. At the beginning [7], it was thought that a Betatron could be indeed the ideal storage ring for the confinement of Crystalline Beams. More recently [8] it was actually determined that the motion of a Crystalline Beam is always unstable, first because the focussing is too weak, and second because of the *shear effect* of the bending magnet.

Configurations

The simplest Crystalline Beam configuration is the *string*, a one-dimensional configuration where particles are located on a common axis, equally spaced by the distance λ . Small perturbations are allowed as long they have an amplitude not larger than a fraction of λ . The next configuration is a two-dimensional structure: the *zig-zag*. It can be thought of as two parallel *strings* separated by a distance $2a$, each with the same longitudinal spacing λ but shifted with respect to each other by $\lambda/2$ (see Figure 4). The ratio of the transverse separation $2a$ to the longitudinal spacing λ depends on the strength of the external restoring forces. The *zig-zag* may be either vertical or horizontal. A three-dimensional configuration is given by the *helix* made of a number of *strings* parallel to each other, all with the same longitudinal spacing λ , and symmetrically located along the

contour of an ellipse. The aspect ratio depends on the magnitude of the transverse components of the external forces. Finally, if the original beam is very intense, several *helices* surrounding each other with the same aspect ratio appear.

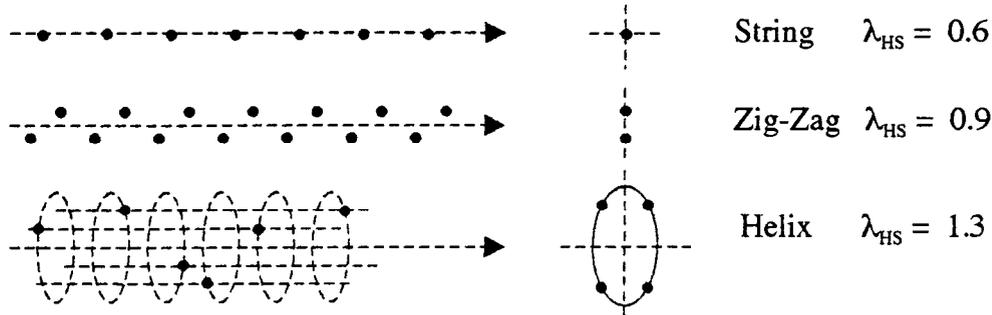


Figure 4. Crystalline Beam Configurations revealed by computer simulation.

The Procedure

To study the formation and the properties of Crystalline Beams in their ground state there is a procedure [11] made essentially of the following four major steps:

- i. One begins by defining a desired Equilibrium Configuration made of *strings*, *zig-zags*, and *helices*. This will depend on the beam parameters like intensity and energy, and on the focussing and bending properties of the storage ring, namely the lattice.
- ii. One then searches for the Confinement Conditions; that is, the requirements on the magnitude and distribution of the external forces to maintain the Equilibrium Configuration. Our interest here is only in continuous beams that do not need longitudinal confinement.
- iii. By adding a small perturbation to the motion of a test particle one derives the Stability Conditions that have also to be fulfilled by the same external forces that provide confinement.
- iv. Finally, one determines the temperature required to achieve the phase transition to the crystalline configuration, and express it in the terms of the beam momentum spread and transverse emittance.

Storage Ring and Beam Parameters

The Storage Ring is essentially described by its circumference $C = 2\pi R$, and the Periodicity of the lattice P which in the following we assume to be even. It is also desirable that each period has an internal mirror symmetry. The motion of a particle is

described by the betatron tunes ν_H and ν_V , and the transition energy γ_T ; and by the amplitude lattice functions β_H and β_V , the curvature h of the reference closed orbit, and the dispersion η which also vary periodically along the circumference of the ring. We assume here that the curvature exists only on the horizontal plane.

On the other end, as we have already seen, a Beam of charged particles is described by the charge state Q and the mass number A of the ions, their velocity and energy relativistic factors β and γ , and the intensity, that is, the total number N of circulating particles. A descriptive parameter is also the average longitudinal spacing $\lambda = N/C$, that we have already seen appearing in the Hasse-Schiffer parameter.

How to get a String

Start with a dilute beam, that is, low intensity N and large spacing λ as shown in Figure 5a. We expect also that the effects of the space-charge forces are small. Apply cooling to reduce the beam temperature by removing internal energy. As the beam temperature drops the amplitude of the transverse oscillations reduces, bringing the particles closer together. As the beam gets denser Space-Charge (SC) forces and Intra-Beam Scattering (IBS) also increase. The cooling rate ought to be sufficiently fast to overcome SC forces and IBS. Eventually, the amplitude of the transverse oscillation becomes comparable to the average spacing λ (Figure 5b), and even smaller (Figure 5c).

At the same time, at the start, the beam momentum spread is large and particles drift longitudinally. Those with larger momentum overtake particles in the middle of the beam, and those with lower momentum will lag behind (Figure 6a). With Cooling also the momentum spread reduces, and when the spread is sufficiently small, particles acquire an aligned configuration, as shown in Figure 6b. The position of each particle ultimately will freeze with spacing λ . They align themselves one behind the other, and ions perform longitudinal and transverse oscillations with amplitude $a \ll \lambda$. We have thus obtained a *string*! [12].

In a *string* configuration, ions perform small-amplitude oscillations, in the longitudinal direction at the frequency $\nu_s f_0$, where f_0 is the revolution frequency, and in the two transverse directions with frequencies $\nu_{H,V} f_0$. In this lowest configuration state, space-charge forces are not very significant, and the betatron tunes $\nu_{H,V}$ are unaffected. On the other end, the longitudinal frequency is about given by Eq. (1), which depends on the particle-to-particle density and thus on the beam intensity.

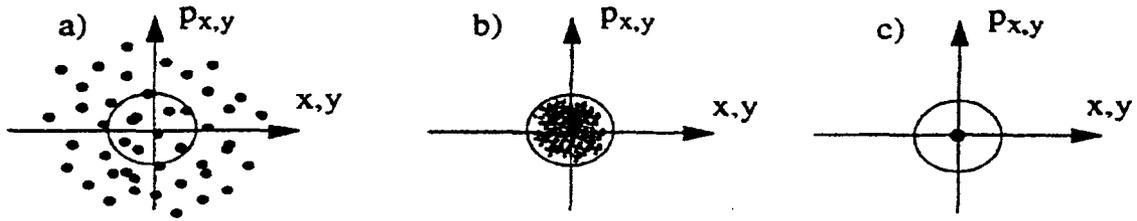


Figure 5. Particle distribution in the transverse phase planes: a) dilute, before cooling; beam dimension is larger than average longitudinal spacing λ , shown by circle with the same radius. b) dense, during cooling; beam dimension is comparable to λ . c) phase transition to the *string*; the amplitude of the transverse oscillations is considerably smaller than λ .

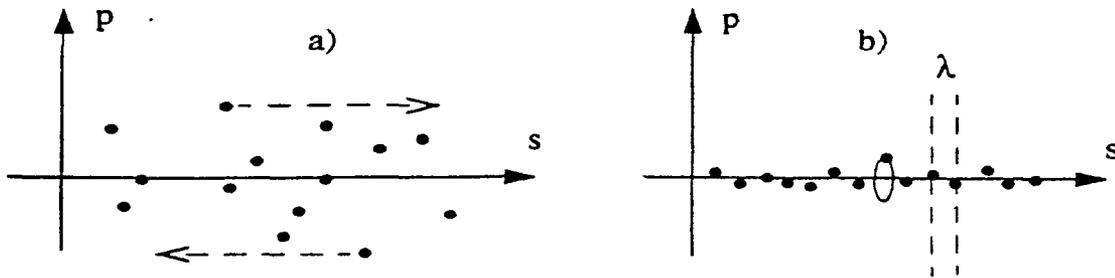


Figure 6. Particle distribution in the longitudinal phase planes: a) dilute, before cooling; because momentum spread is large, particles drift. b) dense, after cooling; particles acquire equal spacing λ and oscillate (as shown by small ellipse).

How to get a Zig-Zag

Once we have obtained a *string*, we can repeat the same conceptual experiment described above. We increase first the beam intensity N and reduce correspondingly the average longitudinal spacing λ . Increasing the beam intensity, of course, will make the space-charge forces more pronounced. They depress the transverse oscillation frequencies, and at the same time raise the longitudinal oscillation frequency. At a certain critical value of λ (that is N), the oscillation frequencies and the revolution frequency f_0 may be found in a parametric resonance. When this occurs, the motion of an individual particle becomes unstable and the *string* configuration undergoes a transformation. The phase-space origin becomes an unstable fixed-point around which the motion diverges. This is accompanied by two new stable fixed points located off-axis, as shown in Figure 7. Particles now oscillate around the new pair of stable fixed points separated by a distance $2b (\neq \lambda)$ which depends on how close is the motion to the parametric resonance. A *zig-zag* is thus formed! [13,14].

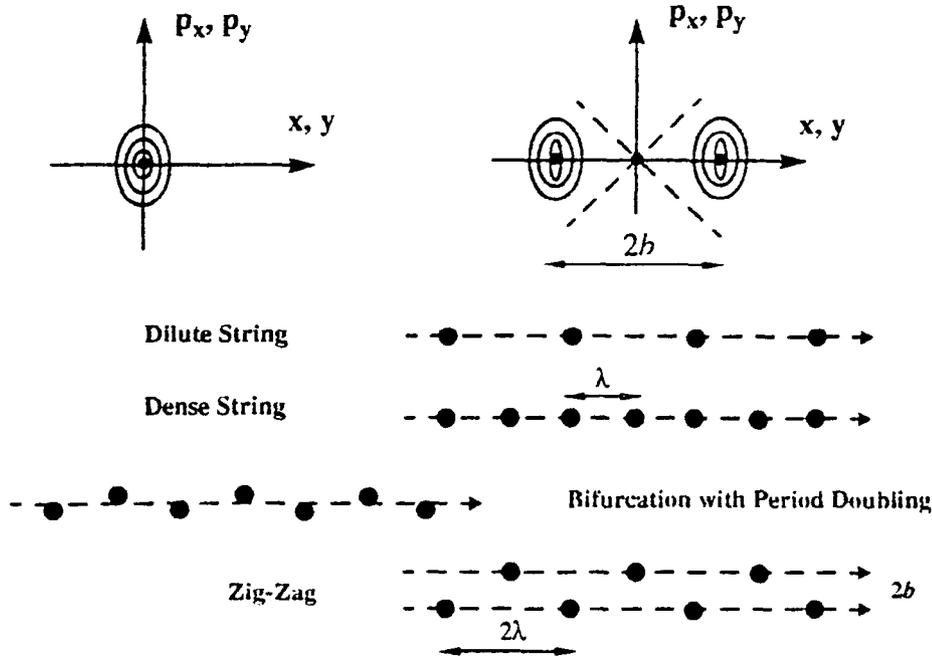


Figure 7. The formation of a Zig-Zag.

Bifurcation and Resonances

The phenomenon just described is called *Bifurcation*, which is always associated to *Period Doubling* [15,16]. A *Bifurcation* occurs either in the radial or vertical direction depending on which parametric resonance is encountered first during the shift of the oscillation frequencies induced by space charge

$$2 \nu_{HV} = m P \quad m = 0, 1, 2, \dots \quad (5)$$

It is also possible that a *momentum bifurcation* occurs if the longitudinal resonance is encountered first

$$2 \nu_s = P. \quad (6)$$

Another possibility is coupling between radial and momentum oscillations

$$\nu_H + \nu_s = m P, \quad m = 0, 1, 2, \dots \quad (7)$$

Longitudinal and coupling resonances cause *bifurcation* only on the radial plane, where trajectories have curvature.

It is to be emphasized that it is the space charge that causes *Bifurcation* to occur, when the tune depression caused by space charge itself causes the approaching and the crossing of a structural resonance of the type (5) or (6) or (7). Imperfection resonances are not relevant. Any configuration can be thought of a number n_s of sub-strings parallel to each other and placed symmetrically around the common axis. The particle separation λ is the same to all sub-strings. The number of sub-strings is a power of 2, that is, $n_s = 2^p$ where $p = p_H + p_V$ is the order of bifurcation, of which p_H appear in the radial direction and p_V in the vertical direction. The critical value of the spacing for a *bifurcation* is

$$\lambda_c = (1.2 Q^2 r_0 R^2 / A \beta^2 \gamma^5)^{1/3} \quad (8)$$

The existence and direction of the *bifurcation* depends on the periodicity P and the tunes ν_{HV} of the storage ring. The process of *bifurcation* and *period doubling* repeats over and over generating more-complex configurations of Crystalline Beams as shown in Figure 8.

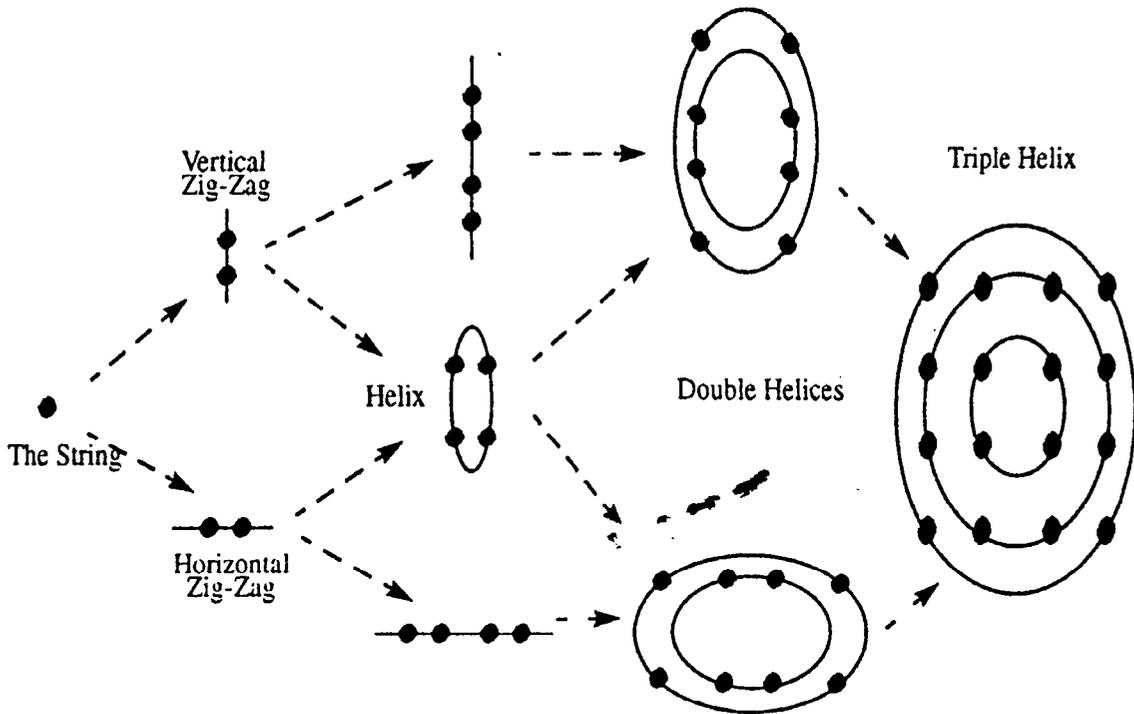


Figure 8. Generation of Crystalline Beam Configurations by *Bifurcation*.

The repetitive application of the conceptual experiment where, keeping all other conditions unchanged, the intensity N increases, from one step to next, will cause the reducing of the spacing λ also for the *zig-zag* until, due to the encounter with one of the resonances described above, a new stability limit is reached.

The approach described here, which explains the formation of Crystalline Beam configurations by determining the effect of the space-charge forces by approaching a low-order stop-band resonance, gives results which are in remarkable agreement with those from other studies where the effect of space charge is directly calculated [17] in a computer program like SYNCH [18] in order to determine the variation of the storage-ring lattice functions.

Equations of Motion

The storage ring setting and the reference closed orbit correspond to a reference momentum value p_0 . A test particle has in general a momentum $p = p_0(1 + \delta)$. Let x and y be respectively the radial and vertical displacements of the trajectory of the test particle from the reference closed orbit. Let also σ denote the difference in path length between the test particle and the particle with reference momentum. We shall use the usual accelerator physics notation, with s the longitudinal coordinate and a prime denoting differentiation with respect to it. The equations of motion then are

$$y'' + K_V(s)y - k_0 F_V(x, y, \sigma) = 0 \quad (9)$$

$$x'' + K_H(s)x - k_0 F_H(x, y, \sigma) = h(s)\delta \quad (10)$$

$$\sigma' = h(s)x - \delta / \gamma^2 \quad (11)$$

$$\delta' = k_0 F_e(x, y, \sigma) \quad (12)$$

where K_H and K_V define the sequence of the focussing elements in the storage ring. They are periodic functions with periodicity P .

$$k_0 = Q r_0 / e A \beta^2 \gamma^3 \quad (13)$$

The components of the particle-particle interaction F_{HVe} can be derived as the components of the gradient of the space-charge potential

$$\mathbf{F}(x, y, \sigma) = -\text{grad} \sum_i Qe / r_i \quad (14)$$

with r_i the distance of the i -th particle in the beam from the test particle. The solution of the system Eq.s (9-12) can be divided in two parts: a particular solution which describes the envelope of the Crystalline Beam in its equilibrium configuration, and a free solution which oscillates around the equilibrium configuration and that can be used to determine the stability of the motion. Let

$$x = x_n + u \quad (15a)$$

$$y = y_n + v \quad (15b)$$

$$\sigma = \sigma_n + \underline{\sigma} \quad (15c)$$

$$\delta = \delta_n + \underline{\delta} \quad (15d)$$

where $x_n, y_n, \sigma_n, \delta_n$ describe the equilibrium configuration of the n -th sub-string, and $u, v, \underline{\sigma}, \underline{\delta}$ are perturbations of motion of a test particle on the same sub-string. Separating the contributions from the two parts and linearizing the particle-particle forces yield the two sets of differential equations: for the Equilibrium Configuration

$$y_n'' + K_V(s) y_n - K_{sc} \zeta_V y_n = 0 \quad (16)$$

$$x_n'' + K_H(s) x_n - K_{sc} \zeta_H x_n = h(s) \delta_n \quad (17)$$

$$\sigma_n' = h(s) x_n - \delta_n / \gamma^2 \quad (18)$$

$$\delta_n' = 2 \gamma^2 K_{sc} \zeta_c \sigma_n \quad (19)$$

and for the Perturbation

$$v'' + K_V(s) v - K_{sc} \eta_V v = 0 \quad (20)$$

$$u'' + K_H(s) u - K_{sc} \eta_H u = h(s) \underline{\delta} \quad (21)$$

$$\underline{\sigma}' = h(s) u - \underline{\delta} / \gamma^2 \quad (22)$$

$$\underline{\delta}' = 2 \gamma^2 K_{sc} \eta_c \underline{\sigma} \quad (23)$$

where

$$K_{sc} = (2/R^2) (\lambda_c / \lambda)^3 = h_{sc}^2 \quad (24)$$

ζ_{HVc} and η_{HVc} are form factors that depend on the arrangement of particles in the Crystalline Beam. For instance, for a single string: $\zeta_{HVc} = 0$ and $\eta_{HVc} = 1$. For a more complex geometry, they are function of the coordinates x_n and y_n . Their definition is summarized in Figure 9. They have been derived in [19]. The two systems of

Eq.s (16-19) and Eq.s (20-23) are written and solved n_s times, each time for a different sub-string. Because of the up-and-down and right-to-left symmetry, only a number equal to the order of *bifurcation* p need to be solved.

Define:

$$\alpha_{nm}^2 = \frac{(x_n - x_m)^2}{\gamma^2 \lambda^2} + \frac{(y_n - y_m)^2}{\gamma^2 \lambda^2} = u_{nm}^2 + v_{nm}^2$$

$$\tau_{nm} = \frac{\sigma_n - \sigma_m}{\lambda}$$

$$f_n(\alpha, \tau) = \sum_i [\alpha^2 + (i - \tau)^2]^{-n/2}$$

$$g_0 = \sum_{i=1}^{\infty} i^{-3} = 1.2$$

Form Factors for Envelope equations:

$$\zeta_h = \frac{1}{g_0} \sum_m f_3(\alpha_{nm}, \tau_{nm}) (x_n - x_m)$$

$$\zeta_v = \frac{1}{g_0} \sum_m f_3(\alpha_{nm}, \tau_{nm}) (y_n - y_m)$$

$$\zeta_e = 2\eta_e \quad (\text{ see below })$$

Form Factors for Stability equations:

$$\eta_h = \frac{1}{2g_0} \sum_m [f_3(\alpha_{nm}, \tau_{nm}) - 3u_{nm}^2 f_5(\alpha_{nm}, \tau_{nm})]$$

$$\eta_v = \frac{1}{2g_0} \sum_m [f_3(\alpha_{nm}, \tau_{nm}) - 3v_{nm}^2 f_5(\alpha_{nm}, \tau_{nm})]$$

$$\eta_e = \frac{1}{2} (\eta_h + \eta_v)$$

Figure 9. Definition of Parameters and Form Factors of Crystalline Beams.

The Shear Effect

It is seen that the coupling between the radial and longitudinal motion is introduced by the product of the parameter K_{sc} with the curvature h in the bending magnet. The ratio $\xi = h_{sc} / h$ can be used to estimate the magnitude of the coupling: $\xi \ll 1$ corresponds to weak coupling; $\xi \sim 1$ corresponds to strong coupling. The vertical motion is de-coupled from the other two components.

The presence of the curvature term h in the equations of motion may have a negative effect on the feasibility of Crystalline Beams in storage rings. We shall see below that indeed this is the case for the special case of Betatron magnets. In the absence of curvature, $h = 0$, the horizontal and vertical motion are de-coupled and all particles have $\delta = \sigma = 0$, that is they are frozen in the longitudinal direction. But in the presence of curvature, particles will acquire different momentum values and have therefore to move around each other longitudinally [19,21]. This effect is called *shear*, mostly represented by Eq. (18). Since x_n will vary periodically around the circumference of the storage ring, it is expected that also δ_n will vary periodically. One allows ions to shift with one another, but the net amount of the *shear* has to vanish periodically. So that horizontally and longitudinally the Crystal will breath periodically. It is obvious that therefore one is interested only in those solutions of the system of Eq.s (16-19) which is periodical in all the four variables involved with the same periodicity of the storage ring. In particular, the solution must satisfy isochronous condition across one period.

The Betatron as a Storage Ring

The simplest storage ring is made of a single 360° bending magnet with constant curvature h and focussing gradient described by the field index n ; i.e., a Betatron where $K_v = nh^2$ and $K_H = (1 - n) h^2$. The question was raised [20,21] whether a Crystalline Beam can be stored in it.

The envelope Eq.s (16-19) can be easily integrated by simply taking x_n and y_n constant. At the same time also the momentum spread $\delta_n = \gamma^2 h x_n$ is constant and $\sigma'_n = 0$. The following confinement conditions are then to be satisfied:

$$nh^2 = K_{sc} \zeta_v(x_n, y_n) \quad (25)$$

$$(1 - n) h^2 = K_{sc} \zeta_H(x_n, y_n) + h^2 \gamma^2 \quad (26)$$

These equations can be solved to determine the coordinates x_n and y_n as function of the index n and curvature h . It is immediately seen that the only possible solution for a *string*, for which $\zeta_{HV} = 0$, requires $h = 0$, which is unphysical.

Moreover, the stability conditions, which can be derived from Eq.s (20-23), are, for the vertical motion

$$n > \xi^2 \eta_v(x_n, y_n) > 0 \quad (27)$$

and for the horizontal motion

$$n < 1 - \gamma^2 - \xi^2 \eta_H(x_n, y_n) < 0 \quad (28)$$

Since in a Betatron $0 < n < 1$ these conditions cannot be satisfied simultaneously. Thus a Crystalline Beam cannot be maintained in a Betatron. On the other end, in an infinitely long transport with no bending, that is $h = 0$, also the horizontal motion is de-coupled from the longitudinal one and Crystalline Beam configurations are possible, all with $\sigma = \delta = 0$, consistent with the finding by Hasse and Schiffer [10].

The Range of Existence

Several methods are available to solve numerically the system of Eq.s (16-19) to determine the envelope of Crystalline Beams. A method makes use of a matrix notation and of convergence iterative procedure [22], for instance by inspecting the eigenvalues of the overall transfer matrix per storage ring period. The exact analytical solution is non-trivial because of the presence of the non-linear space-charge term. Both Eq.s (16 and 17) are similar to the K-V envelope equations [23] without the emittance term because the beam emittance itself is here zero. We shall attempt to describe the solution of the system by using some intuitive and fundamental facts.

We need to estimate two limiting values of the longitudinal spacing, of which the larger one, λ_1 , determines the onset of the configuration being examined made of n_s sub-strings, and the smaller one, λ_2 , determines the stability limit of the same configuration. Thus

$$\lambda_2 < \lambda < \lambda_1 \quad (29)$$

is the range of existence, with the beam undergoing substantial changes entering and leaving it. As λ varies within the range, the actual location (x_n, y_n) of the sub-strings also varies. Of course there is only one beam at any situation and thus only one configuration. It is our “conceptual experiment” that allows us to generate a sequence of beams with increasing intensity, decreasing longitudinal spacing, and varying configuration.

The space charge causes shifting of the oscillation frequencies until a major half-integral stopband of the type given by Eq. s (5-7) is met. This will occur at either λ_1 or λ_2 . For the condition of onset of the configuration we shall apply the method to the envelope Eq.s (16-19), and to determine the stability limit the method is applied to the perturbation Eq.s (20-23). Since the solution of the beam envelope equations is required to be periodic and

closed, the Crystalline Beam configuration can be triggered either by approaching a radial or vertical stopband, that is Eq. (5), but not possibly by the longitudinal and coupling resonances, Eq.s (6 and 7), which may cause loss of stability but do not generate a closed orbit. The configuration under examination has the *order of bifurcation* p . It is the evolution from a previous configuration of order $p - 1$ evolving to one of order $p + 1$.

In good approximation [17], the betatron tune depression caused by space charge is derived by treating the space-charge term K_{sc} as a perturbation; that is, for the envelope equations

$$\begin{aligned}\Delta v_{HV} &= (1/4\pi) \int \beta_{HV} K_{sc} \zeta_{HV} ds \\ &= (\zeta_{HV} / v_{HV}) (\lambda_c / \lambda)^3\end{aligned}\quad (30)$$

and a similar expression for the perturbed equations where ζ_{HV} is replaced by η_{HV} .

Let δv_{HV} be half of the distance of the original betatron tune v_{HV} from the nearest lower integral stopband. By requiring $\Delta v_{HV} = \delta v_{HV}$, we derive the following relation between the form factors and the spacing λ

$$(\lambda / \lambda_c)^3 = \zeta_{HV}(x_n, y_n) c_{HV} \quad (31)$$

where

$$c_{HV} = 1 / (v_{HV} \delta v_{HV}) \quad (32)$$

The number of relations (31) is equal to the *order of bifurcation* p . They can be solved simultaneously to derive the positions (x_n, y_n) of the sub-strings as function of the longitudinal separation λ . In particular, the value $\lambda = \lambda_1$ at which the configuration appears is obtained by requiring that for all n_s sub-strings involved

$$(\lambda_1 / \lambda_c)^3 < \zeta_{HV}(x_n, y_n) c_{HV} \quad (33)$$

On the other end, the stability is lost at $\lambda = \lambda_2$ when

$$(\lambda_2 / \lambda_c)^3 > \eta_{HV}(x_n, y_n) c_{HV} \quad (34)$$

where the locations of the sub-strings were previously determined as functions of λ by means of Eq. (31).

It can be shown that typically $\lambda_1 \sim 2 \lambda_c$ and $\lambda_2 \sim \lambda_c$. At the stability limit $\lambda \sim \lambda_2$, and the transversal separation among sub-strings is comparable to λ_c .

Storage Ring Requirements

Obviously the first requirement is that the storage ring has a periodic sequence with strong and alternating focussing. The two betatron tunes should be about identical, $\nu_H \sim \nu_V$, and the initial beam as “round” as possible, that is the same betatron emittance in both transverse planes of oscillation. The storage ring ought to operate below the transition energy. The periodicity ought to be larger than the betatron tunes, $P \gg \nu_{HV}$. In this case the form factor c_{HV} has the smallest value, since $\delta\nu_{HV} \sim \nu_{HV} / 2$, yielding the smallest longitudinal spacing λ . The lattice of course should be as smooth as possible: an unbroken sequence of FODO cells with internal symmetry. The bending magnets are to be placed uniformly avoiding large discontinuities in the curvature. Finally, for a required configuration with *order of bifurcation* p , it is required that the periodicity $P \gg p$.

The Circular Radio-Frequency Quadrupole Storage Ring

The Circular Radio-Frequency Quadrupole (CRFQ) is a novel concept of storage ring for the accumulation of intense, low-energy beams of light and heavy ions [24]. The new concept is a natural development of the combined features of conventional storage rings and ion traps, and is basically a linear RFQ bent on itself and closed mechanically. Instead of quadrupole magnets, focussing of the particles is provided by the rf field of the device. The focussing period is very short $L = \beta\lambda$, of only few centimeters, where λ here is the rf wavelength. For a sufficiently low energy, there is no need of bending magnets, since the rf field itself is capable to keep the beam on a circular path. Since electrically the device is not closed, it is expected that ion beams can be stored at considerably higher intensities. The space charge limit is reached when the phase advance per period is lowered, for instance, from 90° down to 45° or even lower. Lowering the phase advance down to 0° with enough beam intensity and cooling active, may actually lead to beam *crystallization*. The advantages of the CRFQ are: small beam dimensions, higher beam intensity, and a more compact storage device. Moreover, it satisfies all the requirements needed to confine and to maintain stable complex Crystalline Beam configurations.

Colliding Crystalline Beams

We have determined that Crystalline Beams are described by the critical particle-to-particle spacing λ_c , given by Eq. (8). Actually λ_c^{-3} is a measure of the ion density that can be reached at the limit of crystallization. We should compare this value with the density of an ordinary gaseous ion beam at the space-charge limit. This is given by the space-charge tune depression which, for a de-bunched beam, can be expressed in terms of the particle separation λ_b

$$\Delta v_{HV} \sim \pi Q^2 r_0 R^2 / A \beta^2 \gamma^3 v_{HV} \lambda_b^3 \quad (35)$$

Comparing this equation with Eq. (8) yields

$$(\lambda_b / \lambda_c)^3 \sim \pi \gamma^2 c_{HV} / 1.2 \quad (36)$$

Thus, at very most, the Crystalline Beam density may equal but not exceed that of the original “warm” beam.

If it should be possible to collide head-on two Crystalline Beams, one can take advantage of the particle organization to enhance the luminosity of the collision. Let us assume that the two colliding beams are identical and have the same intensity. Assume also that they circulate in separate and intersecting rings of the same size. The luminosity in the case the two beams are “warm” is

$$L_w = \alpha N^2 f_0 / \pi b^2 \quad (37)$$

where α is the fraction of the ring circumference taken by the collision, f_0 is the revolution frequency, and πb^2 the common cross-section. On the other end, if each Crystalline Beam is made of n_s sub-strings, each with an equivalent cross-section πa^2 , and we assume that we are able to align each pair of counter-moving sub-strings, the luminosity is

$$L_{CB} = \alpha n_s (N/n_s)^2 f_0 / \pi a^2 \quad (38)$$

The luminosity enhancement is then

$$L_{CB} / L_w = (1 / n_s) (b / a)^2 \quad (39)$$

The radius a of a sub-string is considerably smaller than the major radius b , and essentially determined by the noise of the cooling device.

But can one obtain collision between Crystalline Beams? Without destroying them? The answer, at least for the case of two colliding *strings*, seems to be positive. A stable configuration has been found [25] where the two beams move against each other in a “pretzel” configuration as shown in Figure 10. The two beams arrange themselves so that the counter-moving ions avoid each other. The configuration is stable provided that the amplitude of the “pretzel”

$$g > (\beta\lambda / \pi R) s \tag{40}$$

with $2s$ the “pretzel” period. The following scaling ought to be satisfied in order to take advantage of the collision

$$s > a > g > \text{Ion Size} \tag{41}$$

with a the vibration amplitude due to the residual temperature.

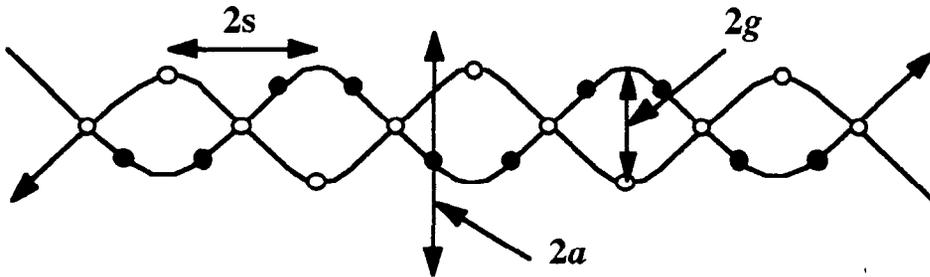


Figure 10. A pair of *Strings* colliding head-on: The “Pretzel”.

Conclusions

Crystalline Beams are a futuristic topic of research. The study of Crystalline Beams is at the front-end of the Accelerator Technology. It helps to understand better the Space Charge and the Intra-Beam Scattering limitations presently encountered in low-energy ions accelerators (including protons). The practical demonstration will provide certainty that Space Charge and Intra-Beam Scattering effects can ultimately be overcome by generating a novel beam configuration. On the sideline, storage ring and laser cooling are

to be expanded with the addition of novel concepts, of which the CRFQ storage ring is an example.

Crystalline Beams are useful for a variety of applications. For instance, with organized formations of ions, it is possible to enhance considerably the luminosity of colliding such beams. At the moment, no applications are foreseen in High-Energy and Nuclear Physics. Possible applications are in the Condensed Matter, Molecular and Atomic Physics. Personally, we are pursuing the concept of using colliding Crystalline Beams to produce nuclear power with the fusion of Boron with Hydrogen.

We find unfortunate and frustrating the poor understanding and the complete lack of support by the funding agencies and institutions for this type of research.

References

- [1] E.P. Wigner, *Trans. Farad. Soc.*, **34**, (1938), 678.
- [2] S.G. Brush, H.L. Sahlin and E. Teller, *The Journal of Chemical Physics*, Vol. **45**, No. 6, (1966), 2102.
- [3] A. Rahman and J.P. Schiffer, *Phys. Rev. Lett.*, **57**, (1986), 1133.
- [4] B.J. Alder and T. Wainright, *The Physical Review*, The First Hundred Years, (1970), 445.
- [5] H. Walther, *Advances in At. Mol. And Opt. Physics*, **31**, (1993), 137.
- [6] See several contributions to the Workshop on Non-Neutral Plasma Physics III, Princeton, New Jersey (1999). AIP Conf. Proc. 498.
- [7] Proceedings of the Workshop on *Crystalline Ion Beams*, Wertheim, Germany, Oct. 1988. GSI-89-10. Editors: R.W. Hasse, I. Hofmann, D. Liesen.
- [8] Proceedings of the Workshop on *Beam Cooling and Related Topics*, Montreux, Switzerland, Oct. 1993. CERN 94-03. Editor: J. Bosser.
- [9] Proceedings of the Workshop on *Crystalline Beams and Related Issues*, Erice, Sicily, Italy, Nov. 1995. Editors: D. Maletic and A.G. Ruggiero. World Scientific.
- [10] R.W. Hasse and J.P. Schiffer, *Ann. of Phys.*, **203**, (1990), 419.
- [11] A.G. Ruggiero, "Crystalline Beams", Workshop on *Quantum-Like Models and Coherent Effects*, Erice, Sicily, Italy, June 1994, page 257. Editors: R. Fedele and P.K. Shukla. World Scientific.
- [12] A.F. Haffmans et al., "Crystalline Beams: The String".
Brookhaven National Laboratory, Internal Report BNL-60619 (1994).
- [13] A.F. Haffmans et al., "Crystalline Beams: The Vertical Zig-Zag".
Brookhaven National Laboratory, Internal Report BNL-60743 (1994).
- [14] A.F. Haffmans et al., "Crystalline Beams: The Horizontal Zig-Zag".

- Brookhaven National Laboratory, Internal Report BNL-60876 (1994).
- [15] A.F. Haffmans et al., “Analytical Studies of Crystalline Beams”, Workshop on *Crystalline Beams and Related Issues*, page 253. Erice, Sicily, Italy, Nov. 1995. Editors: D. Maletic and A.G. Ruggiero. World Scientific.
- [16] A.F. Haffmans, “Confinement and Stability of Crystalline Beams in Storage Rings”, 6th Workshop on *Advanced Accelerator Concepts*, Madison, Wisconsin, June 1994.
- [17] A.F. Haffmans et al., “Study of the Effect of the Space Charge using SYNCH”. Brookhaven National Laboratory, Internal Report BNL-60521 (1994). Also, same Authors, “Behaviour of Space-Charge Dominated Ion Beams in Storage Rings”, Proc. of Particle Accelerator Conference, London, U.K., (1994), 1189.
- [18] A.A. Garren et al., A user’s Guide to SYNCH, FN-240, Fermilab, June 1995.
- [19] A.F. Haffmans et al., “Particle Motion in Crystalline Beams”. Brookhaven National Laboratory, Internal Report BNL-60436 (1994).
- [20] A. G. Ruggiero, “Demonstration of No Feasibility of a Crystalline Beam in a Betatron Magnet”, Brookhaven National Laboratory, Internal Report BNL-49529, (1993).
- [21] X-P Li et al., “Crystalline Beam Ground State”, Brookhaven National Laboratory, Internal Report BNL-52381 (1993).
- [22] A.F. Haffmans et al., “Matrix Formulation of the Particle Motion in Crystalline Beams”, Proc. of Particle Accelerator Conference, London, U.K., (1994), 1382.
- [23] I.M. Kapchinsky and V.V. Vladimirov, Proc. Int. Conf. on High-Energy Accelerators and Instrumentations, CERN 1959, p. 274.
- [24] A. G. Ruggiero, “The Circular RFQ Storage Ring”, Proc. of 1999 Part. Acc. Conf., New York, p. 3731. Also Proc. of the Workshop on NonLinear and Collective Phenomena in Beam Physics, Arcidosso, Italy (1998). Editors: S. Chattopadhyay, M. Cornacchia and C. Pellegrini. AIP conf. proc. 468.
- [25] A.F. Haffmans et al., “Colliding Crystalline Beams”, Proc. of 1995 Part. Acc. Conf., Dallas, Texas, p. 3329.