BNL-65567

Compendium of Equations for the Design of a Very Large Hadron Collider*

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In the following we give several relationships which are used for the preliminary design and to estimate the collider performance. We shall limit to the case of the performance during storage and colliding mode.

Two of such relations are the relation between bending field B, the bending radius ρ and the proton momentum p

$$B\ (\text{Tesla})\ \rho\ (\text{meter}) \ = \ 3.3356\ p\ (\text{GeV/c}) \tag{1}$$

and the minimum requirement of the collider luminosity L which scales with the beam energy E according to

$$L \ = \ (10^{33}\ \text{cm}^{-2}\ \text{s}^{-1})\ (E\ /\ 20\ \text{TeV})^2 \tag{2}$$

We shall assume that the collider is made of two identical intersecting rings where the two beams circulate in opposite directions otherwise with identical configuration, dimensions and intensity. Both beams are bunched. We also assume, for simplicity, that the beams are "round", that is they have the same betatron emittance in the two transverse planes of oscillations, horizontal and vertical. Also the two beams are exactly round at the interaction point where the lattice functions β^* has also the same values in the two planes. Let

N	the number of protons per bunch
M	the total number of bunches per beam
f_0	the revolution frequency at colliding energies
T_0	the revolution period at colliding energies
σ^*	the rms beam size at the collision point
σ_l	the rms bunch length
α	total crossing angle
ε_n	normalized rms betatron emittance
λ	bunch-to-bunch separation
$2\pi R$	collider circumference
β	velocity relativistic factor
γ	energy relativistic factor
γ_T	collider transition energy
$\nu_{H,V}$	number of betatron oscillations per revolution

*Work performed under the auspices of the U.S. Department of Energy

Luminosity and Beam-Beam Tune-Shift

We have, the luminosity

$$L = N^2 f_0 M / 4\pi \sigma^{*2} f \tag{3}$$

and the beam-beam tune-shift

$$\Delta\nu = N r_0 / 2\varepsilon_n (1 + f) \tag{4}$$

where $r_0 = 1.535 \times 10^{-18}$ m is the classical proton radius, and the form factor

$$f = (1 + q^2)^{1/2} \tag{5}$$

with

$$q = \alpha \sigma_l / 2 \sigma^* \tag{6}$$

determines the luminosity reduction due to the crossing angle.

The crossing angle α is needed to avoid that beam bunches circulating in opposite directions, cross with each other also in other locations beside the main interaction point. The following condition then must apply

$$\alpha \lambda / 2 \sigma^* \gg [1 + (\lambda / 2\beta^*)^2]^{1/2} \tag{7}$$

To avoid that the luminosity is reduced unnecessarily by the longitudinal extension of the beam bunches, the following condition also must be satisfied

$$\sigma_l \ll \beta^* \tag{8}$$

On the other end, to avoid enhancement of effects due to betatron synchrotron coupling caused by the modulation of the beam-beam interaction, one needs

$$q < 1 \tag{9}$$

The three conditions (7, 8 and 9) are always satisfied in the discussion that follows.

Microwave Longitudinal Instability

Define the peak current in the proton bunch

$$I_p = N c \beta / \sigma_l (2\pi)^{1/2} \tag{10}$$

then to avoid individual bunch instabilities, it is commonly accepted that one needs to insure that the following stability criterion is satisfied

$$|Z/n| < 3\,E\,|\eta|\,(\sigma_E/E)^2 / e\,I_p \tag{11}$$

where Z/n is the conventionally-defined longitudinal coupling impedance, σ_E/E the rms energy spread, and $\eta = \gamma_T^{-2} - \gamma^{-2}$.

Synchrotron Radiation Effects

Another group of parameters and relationships deals with the effects of the synchrotron radiation. The energy loss per turn is given by

$$U = (7.78 \text{ keV / turn})\, E^4 / \rho \tag{12}$$

where ρ is in meter and E in TeV. The energy damping time is

$$\tau_E = 2\,E\,T_0 / J_E\,U \tag{13}$$

where J_E is the longitudinal repartition factor of the synchrotron radiation. The betatron oscillation factors are denoted below by J_H and J_V. We shall assume that the collider lattice is smooth enough so that $J_E = 2$ and $J_H = J_V = 1$.

At the equilibrium with the synchrotron radiation

$$(\sigma_E/E)^2 = (2.087 \times 10^{-16}\text{ m})\, \gamma^2 / J_E\,\rho \tag{14}$$

From this one can estimate the rms bunch length

$$\sigma_l = c\,|\eta|\,(\sigma_E/E) / 2\pi f_s \tag{15}$$

where f_s is the synchrotron oscillation frequency which obviously depends on the rf choice for the storage mode

$$f_s = f_0\,(h\,|\eta|\,eV \cos\phi_s / 2\pi E)^{1/2} \tag{16}$$

where h is the rf harmonic number, ϕ_s the stable rf phase angle, and V the peak total rf voltage.

The actual rms emittance in equilibrium with the synchrotron radiation is

$$\varepsilon = (J_E\,R / J_H\,\nu_H^3)\,(\sigma_E/E)^2 \tag{17}$$

where ν_H is the number of horizontal betatron oscillations per revolutions. We shall assume that the numbers of betatron oscillations per revolution is the same in the two planes, that is $\nu_V = \nu_H$. We shall also assume that the two modes of oscillations are fully coupled so that the beam has the same emittance value in the two planes of oscillation, that is

$$\varepsilon = \varepsilon_H = \varepsilon_V = \varepsilon / 2 \tag{18}$$

Then, the normalized emittance is related to the actual emittance through the relation

$$\varepsilon_n = \beta \gamma \varepsilon \tag{19}$$

The actual evolution of the beam emittance, for both planes of oscillation, during storage is given by

$$\varepsilon(t) = \varepsilon + (\varepsilon_0 - \varepsilon) \exp(-t/\tau_\beta) \tag{20}$$

where ε_0 is the initial emittance, at the end of the acceleration and just at the beginning of the storage mode. It is very likely determined by the source and injector performance where the effects of the synchrotron radiation are completely negligible. The damping time of the betatron emittance, under these assumptions,

$$\tau_\beta = \tau_E \tag{21}$$