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ON THE DEFINITION OF THE NUMBER OF TEMPORAL MODES IN THE SASE OUTPUT*

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Abstract

The number of temporal modes in the SASE output can be defined in several ways: as the ratio of the phase space area occupied by the radiation to the minimum area allowed by the uncertainty principle; in terms of the fluctuation of the pulse energy; and in terms of the Wigner function. Here, we discuss these different definitions and show their equivalence for SASE from a Gaussian electron bunch, in the linear regime before saturation.

INTRODUCTION

The self-amplified spontaneous-emission free-electron laser (SASE FEL) starts up from the shot noise in the electron beam [1,2]. SASE power increases exponentially as the electrons and radiation co-propagate along the undulator, the exponential gain resulting from a favorable instability build-up between the electron density modulation at the resonant wavelength and the emitted radiation [3-5]. The temporal behavior of the system is that of a narrow-band amplifier with a broadband Poisson seed. Before saturation the output is a Gaussian random process and the radiated field is chaotic, quasi-monochromatic, polarized light. Near saturation, the transverse behavior of the output is dominated by an intense, single spatial mode.

Ignoring the transverse dependence, the radiated electric field can be expressed in the form

$$E(z,t) = A(z,t)\exp(ik_r\, z - i\omega_r\, t), \tag{1}$$

where z represents the location along the undulator at which the SASE is observed and t represents the temporal position in the radiation pulse. For an undulator with period $\lambda_u = 2\pi / k_u$ and magnetic field strength parameter K, the resonant frequency is

$$\omega_r = k_r c = \frac{2k_u\,\gamma^2}{1+K^2/2}. \tag{2}$$

The SASE field before saturation is the superposition of many electromagnetic wave packets emitted from randomly distributed, individual electrons [1,2].

We suppose the electron bunch to have a Gaussian average density profile,

$$D(\tau) = \frac{1}{\sqrt{2\pi}\,\sigma_\tau}\exp\left(-\frac{\tau^2}{2\sigma_\tau^2}\right), \tag{3}$$

where σ_τ is the rms electron bunch duration. We consider the time dependence of the SASE amplitude as observed at a fixed position z. Suppressing the dependence on z, we write the complex, slowly varying amplitude in the form

$$A(t) = A_0 \sum_{j=1}^{Ne} \exp\left(\frac{-\xi\left(t-\tau_j\right)^2}{4\sigma^2} + i\omega_r \tau_j\right). \tag{4}$$

The arrival time τ_j of the jth electron at the undulator entrance is randomly distributed according to the Gaussian distribution of Eq. (3), and

$$\xi = 1 + i\kappa \text{ with } \kappa = 1/\sqrt{3}. \tag{5}$$

It follows from Eq. (4) that the field amplitude $A(t)$ is the sum of independent random variables, so its statistical properties are determined by the Central Limit theorem [6]. This implies that $A(t)$ is a Gaussian random process.

A full treatment of a Gaussian electron bunch would take into account the dependence of the FEL gain on the electron density profile. Here, we ignore this dependence. This allows us to provide an analytic description aimed at illuminating certain qualitative issues. It remains as a challenge to future theoretical work to include the effect of the dependence of gain on the local density, and in particular to determine the temporal duration of the output radiation Σ_t as a function of the electron bunch duration, σ_τ.

MODE NUMBER

Averaging over the stochastic ensemble of arrival times, we determine the field correlation function

$$\langle A(t_1)A^*(t_2)\rangle = \frac{N_e\sigma A_0^2}{\sqrt{\sigma_\tau^2+\sigma^2}}\exp\left(\frac{-\left(1+\kappa^2\right)\sigma_\tau^2(t_1-t_2)^2 - 2\sigma^2\left(\xi t_1^2 + \xi^* t_2^2\right)}{8\left(\sigma_\tau^2+\sigma^2\right)\sigma^2}\right) \tag{6}$$

In deriving Eq. (6), we have retained only the dominant contributions characterized by the absence of rapid phase variation. These correspond to keeping pair-wise equal summation indices from the *A* and *A** terms. It is easily seen that the average of the field vanishes, $\langle A(t)\rangle = 0$.

The Wigner function [7] is defined by

$$W(t,\omega) = \int d\tau \left\langle A\left(t-\frac{\tau}{2}\right)A^*\left(t+\frac{\tau}{2}\right)\right\rangle \exp\left(-i(\omega-\omega_r)\tau\right). \tag{7}$$

From Eqs. (6) and (7), we derive

$$W(t,\omega) = \frac{N_e\sigma A_0^2\sqrt{2\pi}}{\Sigma_t\Sigma_{\omega 0}}\exp\left(-\frac{t^2}{2\Sigma_t^2} - \frac{(\omega-\omega_r-ut)^2}{2\Sigma_{\omega 0}^2}\right), \tag{8}$$

where

$$\Sigma_t^2 = \sigma_\tau^2 + \sigma^2, \quad \Sigma_{\omega 0}^2 = \frac{1+\kappa^2}{4\sigma^2} - \frac{\kappa^2}{4\Sigma_t^2}, \quad u = \frac{\kappa}{2\Sigma_t^2}. \tag{9}$$

Note that for a long pulse, the chirp u [Eq. (9)] vanishes inversely proportional to the square of the pulse duration.

Integrating the Wigner function over frequency, we obtain the average instantaneous intensity,

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$$\left\langle \left|A(t)\right|^2 \right\rangle = \int \frac{d\omega}{2\pi} W(t,\omega) = \frac{N_e \sigma A_0^2}{\Sigma_t} \exp\left(-\frac{t^2}{2\Sigma_t^2} \right). \quad (10)$$

Σ_t is the rms radiation pulse duration. Integrating over time, the average spectral intensity

$$\left\langle \left|\tilde{A}(\omega)\right|^2 \right\rangle = \int dt\, W(t,\omega) = \frac{2\pi N_e \sigma A_0^2}{\Sigma_\omega} \exp\left(-\frac{(\omega-\omega_r)^2}{2\Sigma_\omega^2} \right). \quad (11)$$

The rms radiation bandwidth is given by

$$\Sigma_\omega^2 = \Sigma_{\omega 0}^2 + u^2 \Sigma_t^2 = \frac{1+\kappa^2}{4\sigma^2}. \quad (12)$$

The phase space area occupied by the SASE radiation can be defined as

$$\frac{1}{W(0,0)} \int dt d\omega\, W(t,\omega) = 2\pi\, \Sigma_t \Sigma_{\omega 0}. \quad (13)$$

One definition of mode number, M_1, is the ratio of the phase space area occupied by the radiation to the minimum value allowed by the uncertainty principle. The phase space area (13) occupied by the SASE radiation (see Fig. 1) is proportional to the product, $\Sigma_t \Sigma_{\omega 0}$. The uncertainty principle sets a lower bound of ½ for this quantity. The number of modes in the SASE output can be defined as

$$M_1 \equiv 2\Sigma_t \Sigma_{\omega 0} = \sqrt{1 + \frac{\left(1+\kappa^2\right)\sigma_\tau^2}{\sigma^2}}. \quad (14)$$

A second definition of mode number [8-10], M_2, is given in terms of the fluctuation σ_W of the energy W in a pulse. It is determined by

$$\frac{\sigma_W^2}{W^2} = \frac{\int dt_1 dt_2 \left[\left\langle \left|A(t_1)\right|^2 \left|A(t_2)\right|^2 \right\rangle - \left\langle \left|A(t_1)\right|^2 \right\rangle \left\langle \left|A(t_2)\right|^2 \right\rangle \right]}{\int dt_1 \left\langle \left|A(t_1)\right|^2 \right\rangle \int dt_2 \left\langle \left|A(t_2)\right|^2 \right\rangle} = \frac{1}{M_2} \quad (15)$$

A third definition of the number of modes, M_3 can be given in terms of the Wigner function [7] via,

$$\frac{\int \frac{dt d\omega}{2\pi} W^2(t,\omega)}{\left(\int \frac{dt d\omega}{2\pi} W(t,\omega) \right)^2} = \frac{\int dt_1 dt_2 \left| \left\langle A(t_1) A^*(t_2) \right\rangle \right|^2}{W^2} = \frac{1}{M_3}. \quad (16)$$

The quantity on the left-hand side is often used as a measure of the overall degree of coherence. For a Gaussian random field such as the SASE field introduced in Eq. (4), we know that [9]

$$\left| \left\langle A(t_1) A^*(t_2) \right\rangle \right|^2 = \left\langle \left|A(t_1)\right|^2 \left|A(t_2)\right|^2 \right\rangle - \left\langle \left|A(t_1)\right|^2 \right\rangle \left\langle \left|A(t_2)\right|^2 \right\rangle. \quad (17)$$

In this case, Eq. (15) is equivalent to the definition given in Eq. (16), so

$$M_2 = M_3. \quad (18)$$

For more general fields, where Eq. (17) does not hold, $M_2 \neq M_3$. In this case, the expression in terms of the Wigner function given in Eq. (16) may provide a better definition of the number of modes. In particular, for a fully coherent field with no energy fluctuation, Eq. (16) still holds and correctly states that there is one mode.

For the Gaussian random SASE field (4), using the Wigner function as given in (8) together with Eq. (16), we find that the number of modes is

$$M_3 = M_2 = M_1 = 2\Sigma_t \Sigma_{\omega 0} = \frac{2\sqrt{\pi}\, \Sigma_t}{T_{coh}}, \quad (19)$$

where the coherence time is defined by [8-11]

$$T_{coh} \equiv \int d\tau \left| \frac{\left\langle A\left(t-\frac{\tau}{2}\right) A^*\left(t+\frac{\tau}{2}\right) \right\rangle}{\left\langle \left|A(t)\right|^2 \right\rangle} \right|^2 = \frac{\sqrt{\pi}}{\Sigma_{\omega 0}}. \quad (20)$$

Hence, for the Gaussian random SASE field, we have shown that the number of modes as defined in terms of the energy fluctuation (15) and the Wigner function (16) are equal to the number of minimum area phase space cells occupied by the radiation.

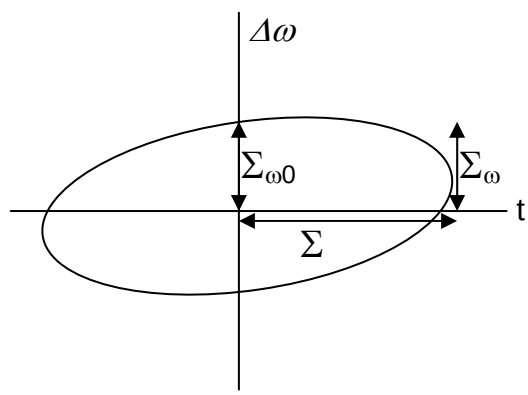

Fig. 1: Region of phase space occupied by radiation, $\Delta\omega = \omega - \omega_r$.

RELATION BETWEEN SASE FROM GAUSSIAN AND TOP-HAT BUNCHES

Let us introduce the scaled field amplitude $a(t)$ via

$$A(t) = A_0 \sqrt{\frac{N_e \sigma}{\Sigma_t}} \exp\left(-\frac{\xi t^2}{4\Sigma_t^2} \right) a(t). \quad (21)$$

It follows from Eqs. (4) and (20) that the scaled field $a(t)$ is the sum of independent random variables, so its statistical properties are determined by the Central Limit theorem. Together with the expression for the correlation given in Eq. (6), this implies that $a(t)$ is a stationary Gaussian random process. Therefore, all the analysis developed by Rice [6] and applied to SASE [9-12] for a uniform distribution can be used to determine the statistics of the scaled field $a(t)$. This in turn determines the statistical properties of the actual field $A(t)$ emitted by a Gaussian bunch.

It is straight-forward to show that $\langle a(t) \rangle = 0$, and it follows from Eqs. (6) and (21) that

$$\langle a(t_1)a^*(t_2)\rangle = \exp\left(-\frac{\sigma_\omega^2(t_1-t_2)^2}{2}\right). \qquad (22)$$

Also,

$$\left\langle |a(t_1)|^2 |a(t_2)|^2 \right\rangle = 1 + \left|\langle a(t_1)a^*(t_2)\rangle\right|^2. \qquad (23)$$

The bandwidth σ_ω is related to the total radiation bandwidth Σ_ω by

$$\Sigma_\omega^2 = \sigma_\omega^2 + \frac{1+\kappa^2}{4\Sigma_t^2} = \frac{1+\kappa^2}{4\sigma^2}. \qquad (24)$$

Recall, σ is the SASE wave packet duration, Eq. (4), and $\kappa = 1/\sqrt{3}$.

One can also write

$$\langle a(t_1)a^*(t_2)\rangle = \int d\Omega\, w(\Omega) e^{-i\Omega(t_1-t_2)}, \qquad (25)$$

where the spectral weight $w(\Omega)$ is given by

$$w(\Omega) = \frac{1}{\sqrt{2\pi}\,\sigma_\omega}\exp\left(-\frac{\Omega^2}{2\sigma_\omega^2}\right). \qquad (26)$$

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